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LETTER TO THE EDITOR

Convergence of self-avoiding random walk to Brownian motion in high dimensions

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Abstract. It is proved that, in sufficiently high dimensions, the scaled self-avoiding random walk on the hypercubic lattice converges in distribution to Brownian motion. Convergence of the finite-dimensional distributions was shown elsewhere. Here tightness is shown.

A T-step self-avoiding random walk ω on the d-dimensional hypercubic lattice $\mathbb{Z}^d$ is a sequence $\omega = (\omega(0) = 0, \omega(1), \ldots, \omega(T))$ with $\omega(i) \in \mathbb{Z}^d$, $|\omega(i+1) - \omega(i)| = 1$ and $\omega(i) \neq \omega(j)$ for $i \neq j$. Equal probability is assigned to each T-step self-avoiding walk.

This model was first introduced by chemists as a model of polymer molecules and has since been studied by physicists as an interesting model of critical phenomena. It is also of interest to probabilitists as a natural example of a non-Markovian process. In terms of rigorous results, very little has been proved about the critical behaviour of the self-avoiding walk in low dimensions. Recently, however, progress has been made with rigorous results in high dimensions, using the lace expansion, which was developed and used in [1] to analyse the weakly self-avoiding walk in five or more dimensions. In [2] the lace expansion was used to show that there is a $d_0 \geqslant 5$ such that for $d \geqslant d_0$ the mean-squared displacement R_T^2 of the T-step step-avoiding walk is asymptotically of the form

$$R_T^2 \sim DT \tag{1}$$

as T approaches infinity, for some constant $D > 1$. In [3] the lace expansion was used to show that, for $d \geqslant d_0$, the number c_T of T-step self-avoiding walks is asymptotically given by

$$c_T \sim \text{constant} \times \beta^T \tag{2}$$

where β is the inverse of the radius of convergence of

$$G(z) = \sum_{T=0}^{\infty} c_T z^T.$$

Also, in [4], the infinite self-avoiding walk was constructed using the lace expansion.

There is at present no good rigorous estimate for the value of d_0, but (1) and (2) are expected to hold for $d \geqslant 5$. For $d = 4$ logarithmic corrections to (1) and (2) are expected, while for $d = 2$ and $d = 3$ it is expected that $R_T^2 \sim \text{constant} \times T^{2\nu}$ and $c_T \sim \text{constant} \times T^{\gamma-1}\beta^T$, with $\nu = \nu(d) > \frac{1}{2}$ and $\gamma = \gamma(d) > 1$. The critical exponents ν and γ have been heuristically and/or numerically computed in all dimensions, but a proof that they have their expected values is lacking for $d < d_0$.

It was also shown in [3], using a variation of the lace expansion, that for $d \geqslant d_0$ the finite-dimensional distributions of $X_n(t)$, defined by

$$X_n(t) = n^{-1/2}\omega([nt]) \tag{3}$$

converge weakly to those of Brownian motion as n approaches infinity. In other words, the expected value of $f(X_n(t_1), X_n(t_2), \ldots, X_n(t_N))$ converges as n approaches infinity to the corresponding Brownian motion expectation, for any $N \geqslant 1$, any bounded continuous function f and any fixed $t_1, \ldots, t_N \in [0, 1]$. The normalisation $n^{-1/2}$ in (3) is the same as the normalisation used in proving that simple random walk (without any self-avoidance constraint) converges in distribution to Brownian motion and is a reflection of the fact that, in high dimensions, self-avoiding walk and simple random walk have essentially the same critical behaviour.

Denote the linear interpolation of X_n by $\tilde{X}_n(t)$. In this letter we prove the following theorem.

Theorem 1. For $d \geqslant d_0$, $\tilde{X}_n$ converges in distribution to Brownian motion as n approaches infinity.

Equivalently, the expected value with respect to the scaled self-avoiding walk of any bounded continuous function on $C_d[0, 1]$ (the $\mathbb{R}^d$-valued continuous functions on the interval $[0, 1]$) converges to the corresponding Brownian motion expectation. This is a stronger result than convergence of the finite-dimensional distributions, since, for example, theorem 1 implies that $\max\{|\tilde{X}_n(t)|: 0 \leqslant t \leqslant 1\}$ (which does not depend on only finitely many times $t_1, \ldots, t_N$) converges in distribution to $\max\{|B_t|: 0 \leqslant t \leqslant 1\}$ as n approaches infinity. Here we consider the Brownian motion to be normalised such that $\langle \exp(ik \cdot B_t)\rangle = \exp(-Dk^2 t/2d)$.

Similar results have been obtained by Lawler [5] for the loop-erased self-avoiding walk (which is the Laplacian random walk with $\eta = 1$ [6]) in four or more dimensions. Lower bounds on the mean-squared displacement critical exponent for the loop-erased walk in two and three dimensions have also been obtained [7].

To prove theorem 1 it suffices to show that the finite-dimensional distributions of $\tilde{X}_n(t)$ converge weakly to those of B_t and that $\{\tilde{X}_n\}$ is tight [8]. Since $|\tilde{X}_n(t) - X_n(t)| \leqslant n^{-1/2}$, convergence of the finite-dimensional distributions of $\tilde{X}_n$ to those of Brownian motion follows from the convergence of the finite-dimensional distributions of X_n proved in [3]. Thus to prove theorem 1 it suffices to show that $\{\tilde{X}_n\}$ is tight. Tightness is a technical condition which amounts to a guarantee that very long excursions by the walk occur with low probability, or more precisely that for each positive ε and η there is a $\delta \in (0, 1)$ and an integer n_0 such that if $n \geqslant n_0$ then

$$\text{Prob}\left\{\max_{|s-t|<\delta} |X_n(t) - X_n(s)| \geqslant \varepsilon\right\} \leqslant \eta.$$

The need for some condition such as tightness can be seen from the following example, which is taken from [8]. Define $x_n \in C_1[0, 1]$ by

$$x_n(t) = \begin{cases} nt & \text{if} \quad 0 \leqslant t \leqslant n^{-1} \\ 2 - nt & \text{if} \quad n^{-1} \leqslant t \leqslant 2n^{-1} \\ 0 & \text{if} \quad 2n^{-1} \leqslant t \leqslant 1 \end{cases}$$

and let P_n be the unit mass measure (or delta function) at x_n. Then the finite-dimensional distributions of P_n converge to those of the unit mass measure at the zero function, but P_n does not converge in distribution to the unit mass at zero since the expected value of $\max\{x(t): 0 \leqslant t \leqslant 1\}$ with respect to P_n is one, for all n.

Tightness is proved via the following lemma. Angular brackets denote expectation with respect to the uniform measure on n-step self-avoiding walks.

Lemma 1. The sequence $\{\tilde{X}_n\}$ is tight if there exist constants $A \geqslant 0$, and $\alpha > \frac{1}{2}$, such that for $0 \leqslant t_1 < t_2 < t_3 \leqslant 1$, and for all n,

$$\langle |X_n(t_2) - X_n(t_1)|^{2\alpha} |X_n(t_3) - X_n(t_2)|^{2\alpha} \rangle \leqslant A|t_2 - t_1|^{\alpha} |t_3 - t_2|^{\alpha}. \tag{4}$$

This lemma, although not stated explicitly in [8], follows in a straightforward manner from results in [8, pp 87-9]. Next we show how (1), (2) and subadditivity can be used to give a simple proof that (4) is satisfied, with $\alpha = 1$, for the self-avoiding walk with $d \geqslant d_0$.

Here we prove that, for $d \geqslant d_0$, (4) is satisfied with $\alpha = 1$, and hence by lemma 1 $\{\tilde{X}_n\}$ is tight and theorem 1 is proved. The assumption that $d \geqslant d_0$ enters the proof in the use of (1) and (2). We begin by introducing some notation. For positive integers $a < b$ we define $\mathscr{B}[a, b]$ to be the set of all pairs of integers (s, t) with $a \leqslant s < t \leqslant b$:

$$\mathscr{B}[a, b] = \{(s, t): a \leqslant s < t \leqslant b;\ s, t \in \mathbb{Z}\}. \tag{5}$$

Elements of $\mathscr{B}[a, b]$ are called bonds. Let

$$U_{s,t}(\omega) = \begin{cases} -1 & \omega(s) = \omega(t) \\ 0 & \omega(s) \neq \omega(t). \end{cases} \tag{6}$$

Define

$$K[a, b] = \prod_{(s,t) \in \mathscr{B}[a,b]} (1 + U_{s,t}(\omega)). \tag{7}$$

Then $K[a, b]$ is equal to 1 for a walk which is self-avoiding and is equal to 0 for a walk which intersects itself, and hence the left-hand side of (4), with $\alpha = 1$, can be written as

$$\langle |X_n(t_2) - X_n(t_1)|^2 |X_n(t_3) - X_n(t_2)|^2 \rangle = n^{-2} c_n^{-1} \sum_{|\omega| = n} |\omega(nt_2) - \omega(nt_1)|^2 |\omega(nt_3) - \omega(nt_2)|^2 K[0, n] \tag{8}$$

where the sum is over all n-step *simple* random walks.

Now subadditivity can be expressed by the inequality

$$K[a, b] \leqslant K[a, c] K[c, b] \tag{9}$$

where c is any integer such that $a \leqslant c \leqslant b$. The inequality (9) follows from the fact that $1 + U_{s,t}(\omega) \leqslant 1$, so omitting bonds (s, t) in (7) with $s < c < t$ gives an upper bound. We use subadditivity in the form

$$K[0, n] \leqslant K[0, nt_1] K[nt_1, nt_2] K[nt_2, nt_3] K[nt_3, n] \tag{10}$$

in (8). This allows the sum over ω to be replaced by a sum over independent subwalks on the time intervals $[0, nt_1]$, $[nt_1, nt_2]$, $[nt_2, nt_3]$ and $[nt_3, n]$. Also, by (2),

$$c_n^{-1} \leqslant \text{constant} \times c_{nt_1}^{-1} c_{nt_2 - nt_1}^{-1} c_{nt_3 - nt_2}^{-1} c_{n - nt_3}^{-1}. \tag{11}$$

When (10) and (11) are substituted in (8), the sum over the subwalk on the interval $[0, nt_1]$ is cancelled by the factor $c_{nt_1}^{-1}$ on the right-hand side of (11) and the sum over the subwalk on the interval $[nt_3, n]$ is cancelled by $c_{n-nt_3}^{-1}$. This yields the estimate

$$\begin{aligned}
\langle |X_n(t_2)-X_n(t_1)|^2 |X_n(t_3)-X_n(t_2)|^2\rangle \\
&\leqslant A_1 n^{-2} c_{nt_2-nt_1}^{-1} \sum_{|\omega|=nt_2-nt_1} |\omega(nt_2-nt_1)|^2 K[0, nt_2-nt_1] \\
&\quad \times c_{nt_3-nt_2}^{-1} \sum_{|\omega|=nt_3-nt_2} |\omega(nt_3-nt_2)|^2 K[0, nt_3-nt_2] \\
&= A_1 n^{-2} \langle |\omega(nt_2-nt_1)|^2\rangle \langle |\omega(nt_3-nt_2)|^2\rangle \qquad (12)
\end{aligned}$$

where the expectations on the right-hand side are with respect to nt_2-nt_1 and nt_3-nt_2 step walks, respectively, and A_1 is a constant. By (1) these expectations are bounded above by a constant multipled by nt_2-nt_1 and nt_3-nt_2, which upon substitution in (12) gives (4) with $\alpha=1$.

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